On a traffic problem: Filippov system formulation for $N$ cars

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Follow-the-leader Model of the Traffic Flow I

- Consider the microscopic follow-the-leader model of the traffic flow on the circular road of length $L$

$$
\begin{align*}
x'_i &= y_i, \\
y'_i &= \frac{1}{\tau} [V(x_{i+1} - x_i) - y_i], \quad x_{N+1} = x_1 + L,
\end{align*}
\tag{1a}
$$

$$
\begin{align*}
i = 1, \ldots, N, \text{ where } N \text{ is a number of cars on the road.}
\end{align*}
\tag{1b}
$$

- $(x_i, y_i)$ – position and velocity of the $i$-th car,
- $\tau$ – reaction time of a driver,
- $V : r \mapsto V(r)$ – optimal velocity (OV) function,

- Choice of $V$ imposes a driving law.
- We consider a hyperbolic OV function

$$
V(r) = V_{max} \frac{\tanh (a(r - 1)) + \tanh a}{1 + \tanh a}.
\tag{2}
$$
The difference

\[ h_i = x_{i+1} - x_i, \quad i = 1, \ldots, N \]

is called headway of the \( i \)-th car.

Given an initial condition \([x^0, y^0] \in \mathbb{R}^N \times \mathbb{R}^N\), the system (1) defines a flow on \( \mathbb{R}^N \times \mathbb{R}^N \)

\[ [x^0, y^0] \mapsto [x(t), y(t)] \equiv \Phi(t, [x^0, y^0]), \quad t \in \mathbb{R}. \]
Follow-the-leader Model of the Traffic Flow III

- Without loss of generality, we may order $x^0$ as

$$s \leq x_1^0 \leq x_2^0 \leq \cdots \leq x_{N-1}^0 \leq x_N^0 \leq L + s,$$

where $s \in \mathbb{R}$ is an arbitrary phase shift.

- We also assume that all initial velocity components are positive,

$$y^0 = (y_1^0, \ldots, y_N^0), \quad y_i^0 > 0, \quad i = 1, \ldots, N.$$
Non Physical Solutions I

- It is well known that solutions of the follow-the-leader model may become non physical.
- The model breaks down at the time instant when two cars collide.

### Collision

The collision occurs at the time $t_E$ for which there exists $k \in \{1, 2, \ldots, N\}$ such that,

$$h_k(t_E) = x_{k+1}(t_E) - x_k(t_E) = 0, \quad y_k(t_E) > y_{k+1}(t_E).$$

By continuity argument, $h_k(t)$ becomes negative for $t_E < t < T$.

### Remark

The inequality $y_k(t_E) > y_{k+1}(t_E)$ holds generically.
Example (Periodic non physical solution)

Consider $N = 3$, $L = 4.56281$, $V^{\text{max}} = 7$, $a = 2$, $\tau = 1$ and

$$x^0 = [0, 3.2189, 3.8664], \quad y^0 = [5.0324, 5.6519, 1.9646].$$
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One can observe that at $t = t_E = 0.2074$,

$$h_2(t_E) = 0, \quad y_2(t_E) > y_3(t_E).$$

The natural interpretation is that at $t = t_E$ the car No 2 is about to overtake the car No 3.
Overtaking Model

1. Given an initial condition, system (1) is solved numerically.
2. Locate the least time $t_E$ for which there is $k \in \{1, \ldots, N\}$ such that $h_k(t_E) = 0$.
3. Create new initial condition by swapping $[x_k(t_E), y_k(t_E)]$ and $[x_{k+1}(t_E), y_{k+1}(t_E)]$.
4. Recursively repeating steps 1–3 we get event sequence $\{t_E(j)\}$.
5. Reconstructing car permutations, the trajectory of each car is assembled from pieces defined on intervals $[t_E(j - 1), t_E(j))$.

The resulting trajectory is piecewise smooth.

Our main interest is to analyze long-time behavior of Overtaking Model.

For given event sequence \( \{t_E(j)\}_{j=1}^{Z} \) we construct a sequence of symbols called event map

\[
G_E = \{[i_s \rightarrow j_s]\}_{s=1}^{Z}.
\]

Car No \( i_k \) overtakes the car No \( j_k \) at \( t = t_E(k) \), \( k = 1, \ldots, Z \).

Event map proves to be useful to identify and classify invariant objects of Overtaking Model, in particular oscillatory patterns.

For \( N = 3 \), we managed to identify five types of oscillatory patterns.
Example (Rotating wave of class 1)

Consider $N = 3$, $L = 3.093725$, $V^{\text{max}} = 7$, $a = 2$, $\tau = 1$ and

$$x^0 = [0, 0.7440, 0.9102], \quad y^0 = [4.7421, 2.4211, 3.6912].$$
Example (Rotating wave of class 1)

Consider $N = 3, L = 3.093725, V^{\text{max}} = 7, a = 2, \tau = 1$ and

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$$x^0 = [0, 0.7440, 0.9102], \quad y^0 = [4.7421, 2.4211, 3.6912].$$

- The trajectory is periodic with period $T = 6.2226$.
- The event map reads

$$G_E = \{[1 \rightarrow 2], [3 \rightarrow 2], [3 \rightarrow 1], [2 \rightarrow 1], [2 \rightarrow 3], [1 \rightarrow 3], \ldots \}.$$

- The event map $G_E$ is periodic with period $p_E = 6$. 
Filippov Systems

Given a partition \( \{ S_\alpha \} \) of the state space \( \mathbb{R}^n \), the Filippov system of differential equations is generally defined by

\[
x' = f^{(\alpha)}(x), \quad x \in S_\alpha.
\]  

(3)

Hence, system (3) has different right-hand sides in different subsets of the phase space. The right-hand side of (3) may be discontinuous in \( x \). In our model, the subsets \( S_\alpha \) are determined in an adaptive way by the current car ordering on the circuit.


Equivalent Formulation of Follow-the-leader Model

Follow-the-leader model in headway–velocity coordinates

\[ h_i' = y_{i+1} - y_i, \quad y_{N+1} = y_1 \]  \hspace{2cm} (4a)

\[ y_i' = \frac{1}{\tau} [V(h_i) - y_i] , \]  \hspace{2cm} (4b)

\( i = 1, \ldots, N \)

System (4) is equivalent to (1) up to a shift in \( x \) variable.

To handle the overtaking without creating new initial conditions at \( t = t_E(j) \) we need to

\begin{itemize}
  \item generalize the notion of headway,
  \item modify the OV function.
\end{itemize}
Definition (Gap variable)

The quantity $h_{i,j}$, $i \neq j$, defined as

$$h_{i,j} = x_j - x_i, \quad i < j, \quad h_{i,j} = L - h_{j,i}, \quad i > j$$

is called gap between car No $i$ and car No $j$.

Proposition

$$h_{i,j} = \sum_{k=i}^{j-1} h_{k,k+1}, \quad i < j, \quad h_{i,j} = L - \sum_{k=j}^{i-1} h_{k,k+1}, \quad i > j$$

(5)

Remark

It is essential that we allow $h_{i,j} < 0$ and $h_{i,j} > L$. 
The driving law imposed by the OV function depends on the current headway of each car, it does not depend on how many laps the leader is ahead / behind.

Therefore, to make the gap variables acceptable as the OV function arguments we need to modify the OV function to be:

- $L$-periodic,
- and consequently discontinuous.

**Definition ($L$-periodic discontinuous OV function)**

$L$-periodic discontinuous OV function $\tilde{V} : r \mapsto \tilde{V}(r)$ is defined via the formula (2) on the interval $[0, L)$ and extended periodically on the whole $\mathbb{R}$. 
Example

Let $L = 3$, $a = 2$, $V^{\text{max}} = 7$. 

![Graph of a discontinuous OV function](image-url)
The Overtaking Model in gap–velocity coordinates reads

\[ h'_{i,i+1} = y_{i+1} - y_i, \quad i = 1, \ldots, N - 1, \quad (6a) \]

\[ y'_i = \frac{1}{\tau} \left[ \bar{V}(h_{i,v(i)}) - y_i \right], \quad i = 1, \ldots, N. \quad (6b) \]

Index \( v(i) \) is a number of the car which is the current leader of the \( i \)-th car. Note that \( h_{i,v(i)} \) is computed by relation (5).
Overtaking Model as a Filippov System V
Detection of overtaking

- Overtaking occurs at the time $t_E$ if there exist $i, j \in \{1, 2, \ldots, N\}$, $i \neq j$, such that $h_{i,j}(t_E) = kL$, $k \in \mathbb{Z}$.
  - If $h'_{i,j}(t_E) < 0$ then the car No $i$ overtakes the car No $j$.
  - If $h'_{i,j}(t_E) > 0$ then the car No $i$ is overtaken by the car No $j$.

- Since any car can overtake only its leader, it is sufficient to look for roots of equations

$$f_i(t) = h_{i,v(i)}(t) - \left[ \frac{h_{i,v(i)}(t_{E}(j))}{L} \right] L = 0, \quad i = 1, \ldots, N,$$

where

$$\lfloor x \rfloor = \max_{n \in \mathbb{Z}} \{ n < x \},$$

$t_{E}(j)$ stands for the last instant at which the overtaking occurred.
Overtaking Model as a Filippov System VI

Update of $v(i)$

- Let the cars are running ordered

  \[ \ldots, p, i, j, q, \ldots \]

Then

\[ v(p) = i, \quad v(i) = j, \quad v(j) = q. \]

- Let $[i \rightarrow j]$ at the time $t_E$.
- For $t > t_E$, the running order is

  \[ \ldots, p, j, i, q, \ldots \]

- Therefore we need to update $v(p)$, $v(i)$ and $v(j)$ as follows

\[ v^{\text{new}}(p) = j = v(i), \quad v^{\text{new}}(i) = q = v(j), \quad v^{\text{new}}(j) = i = v(p) \]
Simulation Procedure I

Initialization

1. Set initial condition

\[ h_{i,i+1}(t_0) = h_{i,i+1}^0 \in \mathbb{R}, \quad i = 1, \ldots, N - 1, \]

\[ y_i(t_0) = y_i^0 > 0, \quad i = 1, \ldots, N. \]

Set \( t_E(0) = t_0, j = 0. \)

2. Set \( x_1 = 0 \) and

\[ x_i = \sum_{k=1}^{i-1} h_{k,k+1}(t_0), \quad i = 2, \ldots, N. \]

3. Compute initial leaders via formula

\[ \nu(i) = \arg \min_{j \neq i \mod (x_j - x_i, L)}. \]
Simulation Procedure II

**Simulation**

1. Integrate system (6) and monitor test functions

   \[ f_i(t) = h_{i,v(i)}(t) - \left[ \frac{h_{i,v(i)}(t_E(j))}{L} \right] L, \quad i = 1, \ldots, N. \]

2. Locate least \( t_E \) for which there is \( k \in \{1, 2, \ldots, N\} \) such that \( f_k(t_E) = 0 \).

3. \( j = j + 1, \quad t_E(j) = t_E, \quad G_E(j) = [k \rightarrow v(k)], \) update \( v(i) \).

4. Repeat steps 1-3.
“Bifurcation Diagram”

\[ N = 4, \ V^\text{max} = 7, \ a = 2, \ \tau = 1, \ L \in [0.72, 7.28] \]
\[ L = 6.4186, \ T = 9.3842, \ p_E = 6 \]
\[ N = 4, \ V^{\text{max}} = 7, \ a = 2, \ \tau = 1 \]
$L = 6.4186, \ T = 9.3842, \ p_E = 6$

$N = 4, \ V^{\text{max}} = 7, \ a = 2, \ \tau = 1$

- Event map:

$$G_E = \{ [2 \rightarrow 3], [1 \rightarrow 3], [1 \rightarrow 2], [3 \rightarrow 2], [3 \rightarrow 1], [2 \rightarrow 1], \ldots \}$$

- Car No 4 neither overtakes nor is being overtaken.

- $G_E$ is identical (up to a shift) to the Event Map corresponding to rotating wave of class 1 for $N = 3$.

- The pattern satisfies

$$y_i(t + T) = y_i(t), \quad i = 1, 2, 3, 4,$$

$$h_{i,j}(t + T) = h_{i,j}(t), \quad i, j = 1, 2, 3, 4, \ i \neq j.$$
\[ L = 1.6477, \quad T = 28.4403, \quad \rho_E = 15 \]

\[ N = 4, \quad V^{\text{max}} = 7, \quad a = 2, \quad \tau = 1 \]
\[ L = 1.6477, \ T = 28.4403, \ \rho_E = 15 \]

\[ N = 4, \ V^{\text{max}} = 7, \ a = 2, \ \tau = 1 \]

- Event map:

\[ G_E = \{[1 \rightarrow 2], [3 \rightarrow 2], [4 \rightarrow 2], [1 \rightarrow 2], [3 \rightarrow 2], [3 \rightarrow 1], [2 \rightarrow 1], [4 \rightarrow 1], [3 \rightarrow 1], [2 \rightarrow 1], [2 \rightarrow 3], [1 \rightarrow 3], [4 \rightarrow 3], [2 \rightarrow 3], [1 \rightarrow 3], \ldots \} \]

- Car No 4 is never overtaken.

- The pattern satisfies

\[ y_i(t + T) = y_i(t), \quad i = 1, 2, 3, 4, \]

\[ h_{i,j}(t + T) = h_{i,j}(t), \quad i, j = 1, 2, 3, \quad i \neq j, \]

\[ h_{4,j}(t + T) = h_{4,j}(t) - L, \quad j = 1, 2, 3, \]

\[ h_{i,4}(t + T) = h_{i,4}(t) + L, \quad i = 1, 2, 3. \]
Conclusion and Outlook

- Formulation of the Overtaking Model as a Filippov system was proposed.
- Preliminary simulation results:
  - several oscillatory patterns identified,
  - construction of the “bifurcation diagram”.
- The main advantage of the Filippov system formulation is the possibility of using standard software (AUTO97, MATCONT, etc.) to continue these patterns with respect to a parameter.