On the problem of uniqueness for the steady Navier–Stokes equation in a cascade of profiles

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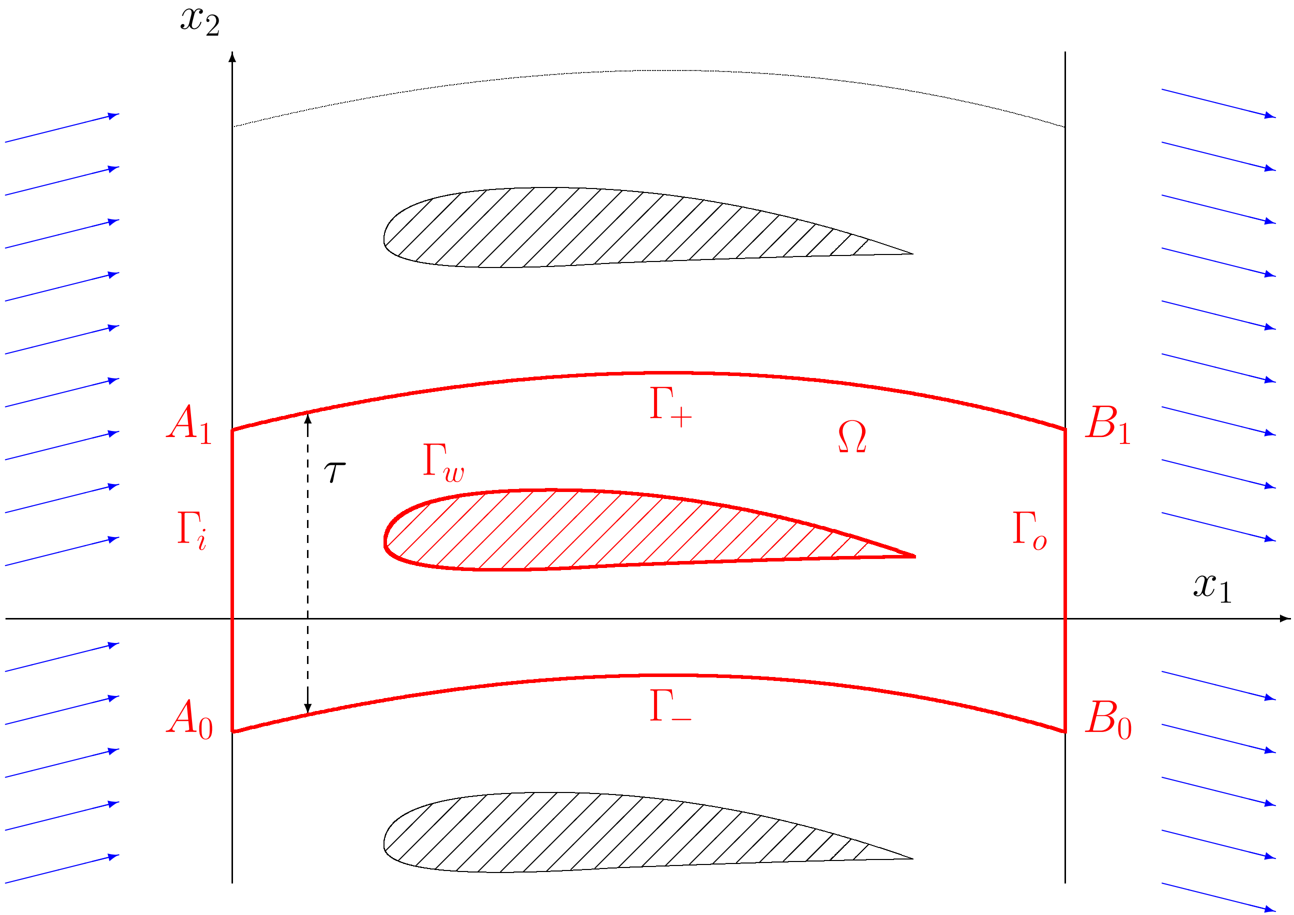
- Geometry of the problem
- The problem with linear boundary conditions
- The problem with a nonlinear boundary condition on the outflow

We consider a flow through the cascade of profiles. We assume that the fluid is **viscous, incompressible, Newtonian** and the flow is **steady** and **2D**.


x_2
x_1
A_1
B_1
A_0
B_0
Γ_+
Γ_-
Γ_i
Γ_o
Γ_w
Ω
τ

1. The problem with linear boundary conditions

1.1. The equations of motion

The conservation of momentum is expressed by the **Navier–Stokes equation**

$$\boxed{\omega(\boldsymbol{u})\,\boldsymbol{u}^{\perp} \;=\; -\nabla q + \nu\,(-\partial_2, \partial_1)\,\omega(\boldsymbol{u}) + \boldsymbol{f}} \qquad (1)$$

where $q = p + \frac{1}{2}\,(u_1^2 + u_2^2)$ is the so-called **Bernoulli pressure**,
$\omega(\boldsymbol{u}) = \partial_1 u_2 - \partial_2 u_1$ is the **vorticity**, and $\boldsymbol{u}^{\perp} = (-u_2, u_1)$.

$\boldsymbol{u} = (u_1, u_2)$	...	**velocity**,
p	...	**pressure**,
$\boldsymbol{f} = (f_1, f_2)$	...	**specific volume force**,
$\nu > 0$	...	**kinematic viscosity**.

The condition of incompressibility is expressed by **the equation of continuity**

$$\boxed{\operatorname{div} \boldsymbol{u} = 0.} \tag{2}$$

It also expresses the conservation of mass.

1.2 Boundary conditions

The **inhomogeneous Dirichlet condition on the inflow:**

$$\boxed{\boldsymbol{u} = \boldsymbol{g}} \qquad \text{on } \Gamma_i \tag{3}$$

where $\boldsymbol{g}$ is a given velocity on Γ_i.

The **conditions of periodicity on Γ_- and Γ_+:**

$$\boldsymbol{u}(x_1, x_2 + \tau) = \boldsymbol{u}(x_1, x_2) \tag{4}$$

$$\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}}(x_1, x_2 + \tau) = -\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}}(x_1, x_2) \qquad \text{for } (x_1, x_2) \in \Gamma_- \tag{5}$$

$$q(x_1, x_2 + \tau) = q(x_1, x_2) \tag{6}$$

The **homogeneous Dirichlet condition on the profile:**

$$\boxed{\boldsymbol{u} = \mathbf{0}} \qquad \text{on} \quad \Gamma_w \tag{7}$$

The **linear "do–nothing" condition on the outflow Γ_o:**

$$\boxed{q = h_1 \qquad\qquad -\nu\,\omega(\boldsymbol{u}) = h_2} \tag{8}$$

1.3 Function spaces and the weak formulation

$H^1(\Omega)$ (respectively $H^1(\Omega)^2$) is the Sobolev space of scalar (respectively vector) functions, defined a.e. in Ω, with the norm $\| \, . \, \|_1$.

$H^s(\Gamma_i)^2$ (for $0 < s < 1$) is the Sobolev–Slobodetski space of vector functions, defined a.e. in Γ_i, with the norm $\| \, . \, \|_{s;\Gamma_i}$.

$$\begin{aligned} V \; = \; \Big\{ & \boldsymbol{v} \in H^1(\Omega)^2; \quad \boldsymbol{v} = \boldsymbol{0} \;\text{ s.v. v } \Gamma_i \cup \Gamma_w, \\ & \boldsymbol{v}(x_1, x_2 + \tau) = \boldsymbol{v}(x_1, x_2) \;\text{ for a.a. } (x_1, x_2) \in \Gamma_-, \\ & \operatorname{div} \boldsymbol{v} = 0 \;\text{ a.e. in } \Omega \Big\}. \end{aligned}$$

(Conditions on Γ_i, Γ_w a Γ_- are fulfilled in the sense of traces.)
The norm in V is

$$|||\boldsymbol{v}||| \; := \; \|\nabla \boldsymbol{v}\|_{L^2(\Omega)^4} \, .$$

Formal derivation of the weak formulation:

In order to derive formally the weak formulation of the problem, we multiply equation (1) by an arbitrary test function $\boldsymbol{v} = (v_1, v_2) \in V$, integrate in Ω and use Green's theorem. We obtain

$$a(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v})_{L^2(\Omega)^2} + b(\boldsymbol{h}, \boldsymbol{v}) \qquad \forall\, \boldsymbol{v} \in V,$$

where

$$a(\boldsymbol{u}, \boldsymbol{v}) := a_1(\boldsymbol{u}, \boldsymbol{v}) + a_2(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}),$$

$$a_1(\boldsymbol{u}, \boldsymbol{v}) := \nu \left(\omega(\boldsymbol{u}), \omega(\boldsymbol{v})\right)_{L^2(\Omega)},$$

$$a_2(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) := \int_\Omega \omega(\boldsymbol{u})\, \boldsymbol{v}^\perp \cdot \boldsymbol{w}\, \mathrm{d}\boldsymbol{x},$$

$$b(\boldsymbol{h}, \boldsymbol{v}) := -\int_{\Gamma_o} \boldsymbol{h} \cdot \boldsymbol{v}\, \mathrm{d}S.$$

The term implying the form of the condition on Γ_o: $\int_{\Gamma_o} [\nu\, \omega(\boldsymbol{u})\, v_2 - q\, v_1]\, \mathrm{d}S$

The weak formulation of the problem (1)–(8).

Definition 1. *Let* $\boldsymbol{g} \in H^s(\Gamma_i)^2$ *(for some* $s \in (\frac{1}{2}, 1])$ *satisfy the condition* $\boldsymbol{g}(A_1) = \boldsymbol{g}(A_0)$. *Let* $\boldsymbol{f} \in L^2(\Omega)^2$ *and* $\boldsymbol{h} \in L^2(\Gamma_o)^2$. *The* **weak solution** *of the problem (1)–(8) is a vector function* $\boldsymbol{u} \in H^1(\Omega)^2$ *which satisfies the identity*

$$a(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v})_0 + b(\boldsymbol{h}, \boldsymbol{v}) \tag{9}$$

for all test functions $\boldsymbol{v} \in V$, *the equation of continuity (3) a.e. in* Ω *and the boundary conditions (3)–(8) (on* Γ_i, Γ_w, Γ_- *and* Γ_+*) in the sense of traces.*

1.4 Existence and uniqueness of a weak solution

Lemma 1. *There exists a constant $c_1 > 0$ independent of $\boldsymbol{g}$ and a divergence–free extension $\boldsymbol{g}^* \in H^1(\Omega)^2$ of function $\boldsymbol{g}$ from Γ_i onto Ω such that $\boldsymbol{g}^* = \boldsymbol{0}$ on Γ_w, $\boldsymbol{g}^*$ satisfies the condition of periodicity*

$$\boldsymbol{g}^*(x_1, x_2 + \tau) = \boldsymbol{g}^*(x_1, x_2) \qquad \text{for } (x_1, x_2) \in \Gamma_- \tag{10}$$

and the estimate

$$\|\boldsymbol{g}^*\|_1 \leq c_1 \|\boldsymbol{g}\|_{s;\Gamma_i} . \tag{11}$$

Now we construct the weak solution $\boldsymbol{u}$ in the form $\boldsymbol{u} = \boldsymbol{g}^* + \boldsymbol{z}$ where $\boldsymbol{z} \in V$ is a new unknown function. Substituting $\boldsymbol{u} = \boldsymbol{g}^* + \boldsymbol{z}$ into equation (9), we get the following problem: *Find a function $\boldsymbol{z} \in V$ such that it satisfies the equation*

$$a(\boldsymbol{g}^* + \boldsymbol{z}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v})_0 + b(\boldsymbol{h}, \boldsymbol{v}) \tag{12}$$

for all $\boldsymbol{v} \in V$.

Theorem (on the existence of a weak solution). *There exists $\varepsilon > 0$ such that if $\|\boldsymbol{g}\|_{s;\Gamma_i} < \varepsilon$ then there exists a solution $\boldsymbol{u}$ of the problem defined in Definition 1.*

Principle of the proof.

We use the Galerkin method and we construct approximations $\boldsymbol{z}_n$ in n–dimensional subspaces V_n of V. The fundamental tool which guarantees the existence of the approximations is the **coerciveness** of the bilinear form a in space V. Applying successively estimates of the forms a_1, a_2 and b, we can derive the next lemma.

Lemma 2. *There exist positive constants c_2 and c_3 such that*

$$\begin{aligned} a(\boldsymbol{g}^* + \boldsymbol{z}, \boldsymbol{z}) \quad &\geq \quad |||\boldsymbol{z}||| \Big(\nu\, |||\boldsymbol{z}||| - \nu\, c_4\, c_5\, \|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} - c_2\, \|\boldsymbol{g}\|^2_{H^s(\Gamma_i)^2} \\ &\quad -c_3\, \|\boldsymbol{g}\|_{H^s(\Gamma_i)^2}\; |||\boldsymbol{z}||| \Big) \end{aligned} \tag{13}$$

Now the coerciveness of the form a follows from (13) and the assumption on a sufficient smallness of $\|\boldsymbol{g}\|_{s;\Gamma_i}$.

Theorem (on the uniqueness of a weak solution). *There exists $R > 0$ such that if $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are two solutions of the problem from Definition 1 such that $\|\nabla \boldsymbol{u}_1\|_{L^2(\Omega)^4} \leq R$ then $\boldsymbol{u}_1 = \boldsymbol{u}_2$.*

Principle of the proof. The structure of the form a_2 implies that

$$a_2(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{v}) = 0, \qquad \text{for all } \boldsymbol{u} \text{ and } \boldsymbol{v} \text{ from } H^1(\Omega)^2.$$

$$a(\boldsymbol{u}_1, \boldsymbol{v}) - a(\boldsymbol{u}_2, \boldsymbol{v}) = 0.$$

$$a_1(\boldsymbol{u}_1 - \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) = \nu \, |||\boldsymbol{u}_1 - \boldsymbol{u}_2|||^2$$

$$\left|a_2(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_1 - \boldsymbol{u}_2) - a_2(\boldsymbol{u}_2, \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2)\right| \leq c_6 \, R \, |||\boldsymbol{u}_1 - \boldsymbol{u}_2|||^2$$

$$\boxed{\nu \, |||\boldsymbol{u}_1 - \boldsymbol{u}_2|||^2 \leq c_6 \, R \, |||\boldsymbol{u}_1 - \boldsymbol{u}_2|||^2.}$$

2. The problem with a nonlinear boundary condition on the outflow

2.1. Equations of motion

The conservation of momentum is expressed by the **Navier–Stokes equation** in the form

$$(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \boldsymbol{f} - \nabla p + \nu \Delta \boldsymbol{u}. \tag{14}$$

The condition of incompressibility is expressed by **the equation of continuity**

$$\operatorname{div} \boldsymbol{u} = 0. \tag{15}$$

2.2 Boundary conditions

Boundary conditions on $\Gamma_i, \Gamma_+, \Gamma_-$ and Γ_w are the same as in Section 1.

The nonlinear condition on the outflow Γ_o:

$$-\nu \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} + p\,\boldsymbol{n} - \frac{1}{2}(\boldsymbol{u}\cdot\boldsymbol{n})^{-}\boldsymbol{u} = \boldsymbol{h}. \tag{16}$$

2.3 The weak formulation

The used integral identity now has the form

$$a(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v})_{L^2(\Omega)^2} + b(\boldsymbol{h}, \boldsymbol{v}) \qquad \forall\, \boldsymbol{v} \in V,$$

where

$$\begin{aligned}
a(\boldsymbol{u}, \boldsymbol{v}) &:= a_1(\boldsymbol{u}, \boldsymbol{v}) + a_2(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) + a_3(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}),\\
a_1(\boldsymbol{u}, \boldsymbol{v}) &:= \nu \int_\Omega \sum_{i,j=1}^{2} \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} \,\mathrm{d}\boldsymbol{x},\\
a_2(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) &:= \int_\Omega \sum_{i,j=1}^{2} u_j \frac{\partial v_i}{\partial x_j} w_i \,\mathrm{d}\boldsymbol{x},\\
a_3(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) &:= \int_{\Gamma_o} \frac{1}{2} (\boldsymbol{u} \cdot \boldsymbol{n})^- \boldsymbol{v} \cdot \boldsymbol{w} \,\mathrm{d}S,\\
b(\boldsymbol{h}, \boldsymbol{v}) &:= -\int_{\Gamma_o} \boldsymbol{h} \cdot \boldsymbol{v} \,\mathrm{d}S.
\end{aligned}$$

The definition of the weak solution is formally identical with Definition 1, we only use form a in the form given on the preceding page.

Definition 2. *Let $\boldsymbol{g} \in H^s(\Gamma_i)^2$ (for some $s \in (\frac{1}{2}, 1]$) satisfy the condition $\boldsymbol{g}(A_1) = \boldsymbol{g}(A_0)$. Let $\boldsymbol{f} \in L^2(\Omega)^2$ and $\boldsymbol{h} \in L^2(\Gamma_o)^2$. The* **weak solution** *of the problem (14), (15), (3)–(7), (16) is a vector function $\boldsymbol{u} \in H^1(\Omega)^2$ which satisfies the identity*

$$a(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v})_0 + b(\boldsymbol{h}, \boldsymbol{v}) \tag{17}$$

for all test functions $\boldsymbol{v} \in V$, the equation of continuity (15) a.e. in Ω and the boundary conditions (3)–(7) (on Γ_i, Γ_w, Γ_- and Γ_+) in the sense of traces.

Theorem (on the existence of a weak solution). *There exists $\varepsilon > 0$ such that if $\|\boldsymbol{g}\|_{s;\Gamma_i} < \varepsilon$ then there exists a solution $\boldsymbol{u}$ of the problem defined in Definition 2.*

Theorem (on the uniqueness of a weak solution). *There exists $R > 0$ such that if $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are two solutions of the problem from Definition 2 such that $\|\nabla \boldsymbol{u}_1\|_{L^2(\Omega)^4} \leq R$ and $\|\nabla \boldsymbol{u}_2\|_{L^2(\Omega)^4} \leq R$ then $\boldsymbol{u}_1 = \boldsymbol{u}_2$.*

Principle of the proof.

$$a(\boldsymbol{u}_1, \boldsymbol{v}) - a(\boldsymbol{u}_2, \boldsymbol{v}) = 0.$$

$$a_1(\boldsymbol{u}_1 - \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) = \nu \, |||\boldsymbol{u}_1 - \boldsymbol{u}_2|||^2$$

$$\left|a_2(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_1 - \boldsymbol{u}_2) - a_2(\boldsymbol{u}_2, \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2)\right| \leq c_7 \, R \, |||\boldsymbol{u}_1 - \boldsymbol{u}_2|||^2$$

$$|a_3(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_1 - \boldsymbol{u}_2) - a_3(\boldsymbol{u}_2, \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2)| \leq c_8 \, R \, |||\boldsymbol{u}_1 - \boldsymbol{u}_2|||^2$$

$$\boxed{\nu \, |||\boldsymbol{u}_1 - \boldsymbol{u}_2|||^2 \leq c_7 + c_8 \, R \, |||\boldsymbol{u}_1 - \boldsymbol{u}_2|||^2.}$$

The derivation of the estimate for a_3 is technically more complicated.