Discontinuous Galerkin method for convection-diffusion problems

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Programy a algoritmy numerické matematiky 14 Dolní Maxov, June 1-6, 2008





- 2 Discretization of the problem
- 3 Numerical analysis
- 4 Application to compressible flow simulations

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Introduction

- **Our aim**: efficient, accurate and robust numerical scheme for the simulation of viscous compressible flows,
- Model problem:
 - scalar nonstationary convection-diffusion equation with nonlinear convection and nonlinear diffusion,
- discontinuous Galerkin finite element method (DGFEM) with NIPG, SIPG or IIPG variant,
- error estimates of DGFEM for nonlinear nonstationary convection–diffusion problems

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Scalar convection-diffusion equation

• Let $\Omega \subset \mathbb{R}^2$, $\partial \Omega = \partial \Omega_D \cup \partial \Omega_N$, $\partial \Omega_D \cap \partial \Omega_N = \emptyset$, $Q_T \equiv \Omega \times (0, T)$, we seek $u : Q_T \to \mathbb{R}$ such that

$$\frac{\partial u}{\partial t} + \nabla \cdot \vec{f}(u) - \nabla \cdot (\mathbb{K}(u)\nabla u) = g \quad \text{in } Q_T, \quad (1)$$

$$u = u_D \quad \text{on } \partial \Omega_D, \ t \in (0, T),$$
 (2)

$$\mathbb{K}(u)\nabla(u)\cdot\vec{n} = g_N \quad \text{on } \partial\Omega_N, \ t \in (0,T), \qquad (3)$$
$$u(x,0) = u^0(x), \quad x \in \Omega, \qquad (4)$$

where: $\vec{f} = (f_1, f_2), f_s \in C^1(\mathbf{R}), s = 1, 2, \mathbb{K}(u)$ are matrices 2x2.

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Triangulations Space discretization

Triangulations



- let T_h , h > 0 be a partition of $\overline{\Omega}$
- $T_h = \{K\}_{K \in T_h}$, K are polygons (convex, nonconvex),
- let $\mathcal{F}_h = \{ \Gamma \}_{\Gamma \in \mathcal{F}_h}$ be a set of all faces of \mathcal{T}_h ,
- we distinguish
 - inner faces \mathcal{F}_h^I ,
 - 'Dirichlet' faces \mathcal{F}_h^D ,
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Notation

Triangulations Space discretization



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Triangulations Space discretization

Spaces of discontinuous functions

• let $s \ge 1$ denote the Sobolev index,

- let $p \ge 1$ polynomial degree,
- over T_h we define:
 - broken Sobolev space

$$H^{s}(\Omega, \mathcal{T}_{h}) = \{v; v|_{K} \in H^{s}(K) \forall K \in \mathcal{T}_{h}\}$$

• the space of piecewise polynomial functions

 $S_{hp} \equiv \{v; v \in L^2(\Omega), v|_K \in P_p(K) \ \forall K \in \mathcal{T}_h\},\$

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Triangulations Space discretization

Example of a function from $S_{hp} \subset H^s(\Omega, \mathcal{T}_h)$



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Triangulations Space discretization

Broken Sobolev spaces, cont.

for $H^{s}(\Omega, \mathcal{T}_{h})$ we define

• the seminorm

$$|\mathbf{v}|_{H^k(\Omega,\mathcal{T}_h)} \equiv \left(\sum_{K\in\mathcal{T}_h} |\mathbf{v}|^2_{H^k(K)}\right)^{1/2}$$

• for $u \in H^1(\Omega, \mathcal{T}_h)$

- $\langle v \rangle_{\Gamma}$ = mean value of v over face Γ ,
- $[v]_{\Gamma} = \text{jump of } v \text{ over face } \Gamma$

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Broken Sobolev spaces, cont.

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Space discretization

- let *u* be a strong (regular) solution,
- we multiply (1) by $v \in H^2(\Omega, \mathcal{T}_h)$,
- integrate over each $K \in \mathcal{T}_h$,
- apply Green's theorem,
- sum over all $K \in \mathcal{T}_h$,
- we include additional terms vanishing for regular solution,
- we obtain the identity

 $\begin{pmatrix} \frac{\partial u}{\partial t}(t), v \end{pmatrix} + a_h(u(t), v) + b_h(u(t), v) + J_h^{\sigma}(u(t), v) \\ = \ell_h(v)(t) \quad \forall v \in H^2(\Omega, \mathcal{T}_h) \; \forall t \in (0, T), (5)$

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Triangulations Space discretization

Diffusive form

• diffusion term: $-\sum_{K \in \mathcal{T}_h} \int_K \nabla \cdot (\mathbb{K}(u) \nabla u) v \, \mathrm{d}x$,

$$\begin{split} g_{h}(u,v) &= \sum_{K \in \mathcal{T}_{h}} \int_{K} \mathbb{K}(u) \nabla u \cdot \nabla v \, \mathrm{d}x \\ &- \sum_{\Gamma \in \mathcal{F}_{h}^{D}} \int_{\Gamma} \langle \mathbb{K}(u) \nabla u \rangle \cdot \vec{n}[v] \mathrm{d}S \\ &+ \eta \sum_{\Gamma \in \mathcal{F}_{h}^{D}} \int_{\Gamma} \langle \mathbb{K}(u) \nabla v \rangle \cdot \vec{n}[u] \mathrm{d}S \end{split}$$

- $\eta = -1$ SIPG formulation,
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Convective form

• convective term ("finite volume approach"):

$$\sum_{K \in \mathcal{T}_h} \int_{K} \nabla \cdot \vec{f}(u) v \, \mathrm{d}x$$

= $-\sum_{K \in \mathcal{T}_h} \int_{K} \vec{f}(u) \cdot \nabla v \, \mathrm{d}x + \sum_{K \in \mathcal{T}_h} \int_{\partial K} \vec{f}(u) \cdot \vec{n} v \, \mathrm{d}S$.
 $\vec{f}(u) \cdot \vec{n}|_{\Gamma} \approx H\left(u|_{\Gamma}^{(L)}, u|_{\Gamma}^{(R)}, \vec{n}_{\Gamma}\right), \quad \Gamma \in \mathcal{F}_h,$

$$\begin{split} b_h(u,v) &= -\sum_{K \in \mathcal{T}_h} \int_K \vec{f}(u) \cdot \nabla v \, \mathrm{d}x \\ &+ \sum_{\Gamma \in \mathcal{F}_h} \int_{\Gamma} H\left(u|_{\Gamma}^{(L)}, u|_{\Gamma}^{(R)}, \vec{n}_{\Gamma} \right) [v]_{\Gamma} \, \mathrm{d}S \end{split}$$

Triangulations Space discretization

Convective form

• convective term ("finite volume approach"):

$$\begin{split} & \sum_{K \in \mathcal{T}_h} \int_{\mathcal{K}} \nabla \cdot \vec{f}(u) \, v \, \mathrm{d}x \\ = & -\sum_{K \in \mathcal{T}_h} \int_{\mathcal{K}} \vec{f}(u) \cdot \nabla v \, \mathrm{d}x + \sum_{K \in \mathcal{T}_h} \int_{\partial \mathcal{K}} \vec{f}(u) \cdot \vec{n} \, v \, \mathrm{d}S. \\ & \vec{f}(u) \cdot \vec{n}|_{\Gamma} \approx \mathcal{H}\left(u|_{\Gamma}^{(L)}, u|_{\Gamma}^{(R)}, \vec{n}_{\Gamma}\right), \quad \Gamma \in \mathcal{F}_h, \end{split}$$

$$b_{h}(u, v) = -\sum_{K \in \mathcal{T}_{h}} \int_{K} \vec{f}(u) \cdot \nabla v \, \mathrm{d}x + \sum_{\Gamma \in \mathcal{F}_{h}} \int_{\Gamma} H\left(u|_{\Gamma}^{(L)}, u|_{\Gamma}^{(R)}, \vec{n}_{\Gamma}\right) [v]_{\Gamma} \, \mathrm{d}S$$

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Triangulations Space discretization

Definition of forms, cont.

Interior and boundary penalty

$$J^{\sigma}_h(u,v) \ = \ \sum_{\Gamma \in \mathcal{F}^I_h} \int_{\Gamma} \sigma[u] \, [v] \, \mathrm{d}S + \sum_{\Gamma \in \mathcal{F}^D_h} \int_{\Gamma} \sigma u \, v \, \mathrm{d}S,$$

$$\sigma_{\Gamma} = \frac{C_W}{d(\Gamma)}, \quad d(\Gamma) \equiv \min(d(K_{\Gamma}^{(L)}), d(K_{\Gamma}^{(R)})), \quad d(K) \equiv \frac{h_K}{p_K^2}$$

Right-hand-side

$$\ell_{h}(v)(t) = \int_{\Omega} g(t) v \, \mathrm{d}x + \sum_{\Gamma \in \mathcal{F}_{h}^{N}} \int_{\Gamma} g_{N}(t) v \, \mathrm{d}S$$
$$+ \sum_{\Gamma \in \mathcal{F}_{h}^{D}} \int_{\Gamma} (\eta \mathbb{K}(u) \nabla v \cdot \vec{n} \, u_{D}(t) + \sigma \, u_{D}(t) \, v) \, \mathrm{d}S$$
Triangulations Space discretization

Definition of forms, cont.

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Triangulations Space discretization

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Triangulations Space discretization

Semi-discrete variant

• For $u(t,x) \in C^1(0,T; H^2(\Omega))$, we have identity

 $\begin{pmatrix} \frac{\partial u}{\partial t}(t), v \end{pmatrix} + a_h(u(t), v) + b_h(u(t), v) + J_h^{\sigma}(u(t), v) \\ = \ell_h(v)(t), \quad v \in H^2(\Omega, \mathcal{T}_h), \ t \in (0, T), (6)$

(6) makes sense also for u ∈ H²(Ω, T_h).
since S_{hp} ⊂ H²(Ω, T_h), identity (6) makes sense for u, v ∈ S_{hp}

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- (6) makes sense also for $u \in H^2(\Omega, \mathcal{T}_h)$.
- since $S_{hp} \subset H^2(\Omega, \mathcal{T}_h)$, identity (6) makes sense for $u, v \in S_{hp}$

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Triangulations Space discretization

Semi-discrete solution

Definition

We say that u_h is a DGFE solution iff

a)
$$u_h \in C^1(0, T; S_{hp}),$$

b) $\left(\frac{\partial u_h(t)}{\partial u_h(t)} + b_t(u_t(t), u_t) + c \right)$

$$\left(\frac{\partial u_h(t)}{\partial t}, v_h \right) + b_h(u_h(t), v_h) + a_h(u_h(t), v_h) + J_h^{\sigma}(u_h(t), v_h) = \ell_h(v_h)(t) \qquad \forall v_h \in S_{hp}, \ t \in (0, T)$$

c) $u_h(0) = u_h^0$,

system of ODEs,

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Interior penalty Error estimates Numerical example

Interior penalty (1)

penalty form

$$J_h^{\sigma}(u,v) = \sum_{\Gamma \in \mathcal{F}_h^{I\!D}} \frac{C_W}{d(\Gamma)} \int_{\Gamma} [u] [v] \, \mathrm{d}S,$$

- $J_h^{\sigma}(u, v)$ "replace" inter-element continuity,
- $J_h^{\sigma}(u, v)$ ensures the coercivity, i.e, $\exists c > 0$

 $a_h(v,v) + J_h^{\sigma}(v,v) \ge c |||v|||^2, \quad |||v|||^2 \equiv |v|_{H^1(\Omega,\mathcal{T}_h)}^2 + J_h^{\sigma}(v,v),$

• choice of C_W ?

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Interior penalty Error estimates Numerical example

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Interior penalty Error estimates Numerical example

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• choice of C_W?

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Interior penalty Error estimates Numerical example

Interior penalty (2) - choice of C_W

• NIPG: $C_W > 0$ is sufficient since

$a_h(v,v) \geq c_1 |v|^2_{H^1(\Omega,\mathcal{T}_h)},$

• SIPG: $C_W \ge C_W$,

linear diffusion: [Dolejší, Feistauer, NFAO 2005],
non-linear diffusion: [Dolejší, JCAM online 2007],

• IIPG: $C_W \geq \widetilde{C_W}/4$.

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Interior penalty Error estimates Numerical example

Interior penalty (2) - choice of C_W

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Interior penalty Error estimates Numerical example

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Interior penalty Error estimates Numerical example

Error estimates for space semi-discretization - assumptions

 non-linear diffusion: -∇ · (𝔅(𝑢) · ∇𝑢), where 𝔅(𝑢) = {k_{ij}(𝑢)}²_{i,j=1} satisfy:

 k_{ij}(𝑢) : 𝑘 → 𝑘, such that |k_{ij}(𝑢)| < 𝔅_𝔅 < ∞, i, j = 1, 2,
 k_{ij}(𝑢) is Lipschitz continuous for i, j = 1, 2,
 ξ^T𝔅(𝑢)ξ ≥ 𝔅_𝔅 ||ξ||², 𝔅_𝔅 > 0, ξ ∈ 𝑘²

 𝑢 is sufficient regular:

 𝑢 ∈ 𝔅²(0, 𝔅; 𝑘^{s+1}), ∂𝑢/∂𝔅 ∈ 𝔅²(0, 𝔅; 𝑘^s), 𝔅 ≥ 1,
 ||∇𝑢(𝔅)| ≤ 𝔅_𝔅 for a. a. 𝔅 ∈ (0, 𝔅)

- mesh is regular and locally quasi-uniform,
- $u_h \in S_{hp}, \ p \ge 1, \ \mu = \min(p+1, s)$

- sub-optimal in the L^2 -norm, i.e., $O(h^{\mu-1})$,
- optimal in the H^1 -seminorm, i.e., $O(h^{\mu-1})$

Interior penalty Error estimates Numerical example

Error estimates for space semi-discretization - assumptions

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Interior penalty Error estimates Numerical example

Numerical example (1)

$$\frac{\partial u}{\partial t} + \sum_{s=1}^{2} u \frac{\partial u}{\partial x_s} - \varepsilon \Delta u = g \quad \text{in} \quad Q_T = [-1, 1]^2 \times (0, T)$$

- nonlinear $f_1(u) = f_2(u) = u^2/2$ and linear $\mathbb{K}(u) = \varepsilon \mathbb{I}$,
- numerical flux:

$$H\left(u|_{\Gamma}^{(L)}, u|_{\Gamma}^{(R)}, \vec{n}_{\Gamma}\right) = \begin{cases} \sum_{s=1}^{2} f_{s}(u|_{\Gamma}^{(L)}) n_{s}, \text{if } A > 0\\ \sum_{s=1}^{2} f_{s}(u|_{\Gamma}^{(R)}) n_{s}, \text{if } A \le 0 \end{cases},$$

where $A = \sum_{s=1}^{2} f'_{s}(\langle u \rangle) n_{s}$,

• exact solution:

$$u(x, y, t) = (1 - x^2)^2 (1 - y^2)^2 \left(1 - \frac{e^{-t}}{2}\right)^2$$

• mesh with "hanging nodes", P1 approximation, SIPG variant

Interior penalty Error estimates Numerical example

Numerical example (1)

$$\frac{\partial u}{\partial t} + \sum_{s=1}^{2} u \frac{\partial u}{\partial x_s} - \varepsilon \Delta u = g \quad \text{in} \quad Q_T = [-1, 1]^2 \times (0, T)$$

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Interior penalty Error estimates Numerical example

Numerical example (2)

- experimental orders of convergence (EOC)
- EOC is optimal in L^2 -norm, i.e. $O(h^2)$ for P_1 approximation

			<i>t</i> = 4.0		$t ightarrow \infty$	
Ι	$\#T_{h_l}$	h _l	e_h	α_I	e_h	α_l
1	136	2.795E-01	1.6599E-02	-	7.0934E-02	-
2	253	2.033E-01	8.3203E-03	2.169	3.0605E-02	2.640
3	528	1.398E-01	3.8102E-03	2.084	1.1299E-02	2.659
4	1081	9.772E-02	1.8194E-03	2.037	5.7693E-03	1.852
5	2080	6.988E-02	9.1509E-04	2.081	3.0657E-03	1.915
6	4095	4.969E-02	4.7598E-04	1.917	1.4538E-03	2.188
$\overline{\alpha}$				2.059		2.214

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Interior penalty Error estimates Numerical example

Numerical example (3)

• steady-state solution



J. Hozman DGM for convection-diffusion problems

Navier-Stokes equations DGFEM for the Navier-Stokes equations NACA 0012 – steady flow

Navier-Stokes equations

$$\frac{\partial \mathbf{w}}{\partial t} + \sum_{s=1}^{2} \frac{\partial}{\partial x_{s}} \mathbf{f}_{s}(\mathbf{w}) = \sum_{s=1}^{2} \frac{\partial}{\partial x_{s}} \left(\sum_{k=1}^{2} \mathbf{K}_{sk}(\mathbf{w}) \frac{\partial \mathbf{w}}{\partial x_{k}} \right), \quad (7)$$

where

- w : $\Omega \times (0, T) \rightarrow I\!\!R^4$,
- inviscid terms $\mathbf{f}_s: \mathbf{R}^4 \to \mathbf{R}^4, \ s = 1, 2,$
- viscous terms $\mathbf{K}_{sk}: \mathbf{R}^4 \to \mathbf{R}^{4 \times 4}, \ s, k = 1, 2,$
- state equation for perfect gas and relation for total energy,
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Vavier-Stokes equations DGFEM for the Navier-Stokes equations VACA 0012 – steady flow

DGFEM for the Navier-Stokes equations

Space semi-discretization

- inviscid terms: finite volume approach
- viscous terms: SIPG, NIPG, IIPG techniques
- interior and boundary penalty: heuristic choice of C_W .

Other aspects

- semi-implicit time discretization,
- unconditionally stable higher order scheme,
- GMRES solver for linear system at each time step

Vavier-Stokes equations DGFEM for the Navier-Stokes equations NACA 0012 – steady flow

DGFEM for the Navier-Stokes equations

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Navier-Stokes equations DGFEM for the Navier-Stokes equations NACA 0012 – steady flow

NACA 0012 profile – steady flow(1)

- steady non-symmetric laminar flow around the NACA0012 ($M = 0.5, \alpha = 2.0^{\circ}, Re = 5000$)
- adaptive refined mesh



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Navier-Stokes equations DGFEM for the Navier-Stokes equations NACA 0012 – steady flow

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Navier-Stokes equations DGFEM for the Navier-Stokes equations NACA 0012 – steady flow

NACA 0012 profile – steady flow(2)

- semi-implicit scheme, $P_1 P_3$ approximation, adaptive BDF scheme
- SIPG, IIPG, NIPG variant of DGFEM
- drag and lift coefficients (comparison with DLR, VKI)

P_2	CD	CL
SIPG	0.05519	0.04509
IIPG	0.05518	0.04486
NIPG	0.05518	0.04499
DLR	0.05692	0.04487
VKI	0.05609	0.03746

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Navier-Stokes equations DGFEM for the Navier-Stokes equations NACA 0012 – steady flow

NACA 0012 profile – steady flow(3)

• Mach number isolines





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Navier-Stokes equations DGFEM for the Navier-Stokes equations NACA 0012 – steady flow

Mach number distribution, $t \to \infty$



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Pressure distribution, $t ightarrow \infty$



Navier-Stokes equations DGFEM for the Navier-Stokes equations NACA 0012 – steady flow

Thank you for your attention