Linear stability of Euler equations in cylidrical domain

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Flow equations

- inviscid incompressible flow
- cylindrical coordinate system (r, φ, z) rotating about z-axis with angular velocity Ω
- Euler equations

$$\begin{aligned}
\frac{\mathrm{d}w_r}{\mathrm{d}t} - \frac{w_\varphi^2}{r} - 2\Omega w_\varphi - \Omega^2 r + \frac{1}{\varrho}\frac{\partial p}{\partial r} &= 0 \\
\frac{\mathrm{d}w_\varphi}{\mathrm{d}t} + \frac{w_r w_\varphi}{r} + 2\Omega w_r + \frac{1}{\varrho r}\frac{\partial p}{\partial \varphi} &= 0 \qquad (1.1) \\
\frac{\mathrm{d}w_z}{\mathrm{d}t} + \frac{1}{\varrho}\frac{\partial p}{\partial z} &= 0
\end{aligned}$$

- w_r, w_φ,w_z radial, circumferential and axial velocity
- p pressure
- ϱ density

- material derivative

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + w_r \frac{\partial}{\partial r} + \frac{w_\varphi}{r} \frac{\partial}{\partial \varphi} + w_z \frac{\partial}{\partial z}$$

- continuity equation

$$\frac{1}{r} \frac{\partial}{\partial r}(r w_r) + \frac{1}{r} \frac{\partial w_\varphi}{\partial \varphi} + \frac{\partial w_z}{\partial z} = 0 \tag{1.2}$$

- solution domain

$$Q = \{(r, \varphi, z) \,|\, R_1 < r < R_2, 0 \leq \varphi < 2\pi, 0 < z < L\}$$

- base flow with axial symmetry

$$w_{0r}(r,z)\,, \quad w_{0\varphi}(r,z)\,, \quad w_{0z}(r,z)\,, \quad p_0(r,z)$$

- boundary conditions
    - inflow region S_1, $z = 0$: w_{0r}, $w_{0\varphi}$, w_{0z}
    - outflow region S_2, $z = L$: p_0
    - free surface Γ_1, $r = R_1$: p_0 or *surface tension conditions*
    - fixed wall Γ_2, $r = R_2$: w_{0r}

Linear stability

- perturbed state

$$(w_r, w_\varphi, w_z, p) = (w_{0r}, w_{0\varphi}, w_{0z}, p_0) + \varepsilon(v_r, v_\varphi, v_z, \sigma) \tag{2.1}$$

- small parameter ε
- disturbances

$$v_r = v_r(r, \varphi, z, t), \qquad v_\varphi = v_\varphi(r, \varphi, z, t)$$

$$v_z = v_z(r, \varphi, z, t), \qquad \sigma = \sigma(r, \varphi, z, t)$$

- stability

$$(w_r, w_\varphi, w_z, p) \to (w_{0r}, w_{0\varphi}, w_{0z}, p_0) \quad \text{for} \quad t \to \infty$$

- linear stability: insert (2.1) into (1.1), (1.2)

  neglect terms with ε^2

- linearized Euler equations

$$\frac{\partial v_r}{\partial t} + w_{0r}\frac{\partial v_r}{\partial r} + \frac{w_{0\varphi}}{r}\frac{\partial v_r}{\partial \varphi} + w_{0z}\frac{\partial v_r}{\partial z} + \frac{\partial w_{0r}}{\partial r}v_r + \frac{\partial w_{0r}}{\partial z}v_z -$$

$$- \frac{2}{r}w_{0\varphi}v_\varphi - 2\Omega v_\varphi + \frac{1}{\varrho}\frac{\partial \sigma}{\partial r} = 0\,,$$

$$\frac{\partial v_\varphi}{\partial t} + w_{0r}\frac{\partial v_\varphi}{\partial r} + \frac{w_{0\varphi}}{r}\frac{\partial v_\varphi}{\partial \varphi} + w_{0z}\frac{\partial v_\varphi}{\partial z} + \frac{\partial w_{0\varphi}}{\partial r}v_r + \frac{\partial w_{0\varphi}}{\partial z}v_z +$$

$$+ \frac{w_{0r}}{r}v_\varphi + \frac{w_{0\varphi}}{r}v_r + 2\Omega v_r + \frac{1}{\varrho r}\frac{\partial \sigma}{\partial \varphi} = 0\,,$$

$$\frac{\partial v_z}{\partial t} + w_{0r}\frac{\partial v_z}{\partial r} + \frac{w_{0\varphi}}{r}\frac{\partial v_z}{\partial \varphi} + w_{0z}\frac{\partial v_z}{\partial z} + \frac{\partial w_{0z}}{\partial r}v_r + \frac{\partial w_{0z}}{\partial z}v_z + \frac{1}{\varrho}\frac{\partial \sigma}{\partial z} = 0$$

- continuity equation

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\varphi}{\partial \varphi} + \frac{\partial v_z}{\partial z} = 0$$

- boundary conditions on S_1, S_2, Γ_2

$$\begin{array}{lcl} v_r = v_\varphi = v_z = 0 & & S_1 \\ \sigma = 0 & \text{on} & S_2 \\ v_r = 0 & & \Gamma_2 \end{array}$$

- boundary condition on Γ_1
  a) in case without surface tension

$$\sigma = 0 \qquad \text{on} \qquad \Gamma_1$$

b) in case with surface tension

$\Delta(\varphi, z, t)$ ... radial displacement of Γ_1

σ_p ... surface tension coefficient

- impermeability equation

$$v_r = \frac{\partial \Delta}{\partial t} + w_{0z} \frac{\partial \Delta}{\partial z} \qquad \text{on } \Gamma_1$$

- Young-Laplace equation

$$\sigma = \sigma_p \left(\frac{\partial^2 \Delta}{\partial z^2} + \frac{1}{R_1^2} \frac{\partial^2 \Delta}{\partial \varphi^2} \right) \qquad \text{on } \Gamma_1$$

- initial condition

$$\Delta = 0 \qquad \text{on} \quad \bar{\Gamma}_1 \cap \bar{S}_1$$

- stability $\iff (v_r, v_\varphi, v_z, \sigma, \Delta) \to (0, 0, 0, 0, 0)$ for $t \to \infty$

Eigenvalue problem

- modal analysis: separation of varibles t, φ taking normal modes

$$v_r(r,\varphi,z,t) = e^{\lambda t + In\varphi} u_r(r,z)$$
$$v_\varphi(r,\varphi,z,t) = e^{\lambda t + In\varphi} u_\varphi(r,z)$$
$$v_z(r,\varphi,z,t) = e^{\lambda t + In\varphi} u_z(r,z)$$
$$\sigma(r,\varphi,z,t) = e^{\lambda t + In\varphi} h(r,z)$$
$$\Delta(\varphi,z,t) = e^{\lambda t + In\varphi} \delta(z)$$

$I = \sqrt{-1}$... imaginary unit

$\lambda \in \mathbb{C}$... eigenvalue

$n \in \mathbb{N}_0 = \{0, 1, 2, ...\}$... azimutal wave number

$u_r, u_\varphi, u_z, h : D \mapsto \mathbb{C}$, $\delta : \langle 0, L\rangle \mapsto \mathbb{C}$ eigenfunctions (normal modes)

$D = \{(r,z)\,|\,R_1 \le r \le R_2, 0 \le z \le L\}$

- eigenvalue problem

$$\lambda u_r + w_{0r}\frac{\partial u_r}{\partial r} + w_{0\varphi}\frac{nI}{r}u_r + w_{0z}\frac{\partial u_r}{\partial z} + \frac{\partial w_{0r}}{\partial r}u_r + \frac{\partial w_{0r}}{\partial z}u_z -$$

$$-\frac{2}{r}w_{0\varphi}u_\varphi - 2\Omega u_\varphi + \frac{1}{\varrho}\frac{\partial h}{\partial r} = 0$$

$$\lambda u_\varphi + w_{0r}\frac{\partial u_\varphi}{\partial r} + \frac{nI}{r}w_{0\varphi}u_\varphi + w_{0z}\frac{\partial u_\varphi}{\partial z} + \frac{\partial w_{0\varphi}}{\partial r}u_r + \frac{\partial w_{0\varphi}}{\partial z}u_z +$$

$$+\frac{w_{0r}}{r}u_\varphi + \frac{w_{0\varphi}}{r}u_r + 2\Omega u_r + \frac{nI}{\varrho r}h = 0$$

$$\lambda u_z + w_{0r}\frac{\partial u_z}{\partial r} + \frac{nI}{r}w_{0\varphi}u_z + w_{0z}\frac{\partial u_z}{\partial z} + \frac{\partial w_{0z}}{\partial r}u_r + \frac{\partial w_{0z}}{\partial z}u_z + \frac{1}{\varrho}\frac{\partial h}{\partial z} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{nI}{r}u_\varphi + \frac{\partial u_z}{\partial z} = 0$$

- in case with surface tension

$$\lambda\delta + w_{0z}(R_1, z)\delta' - u_r(R_1, z) = 0$$

- boundary conditions

$$S_1 \cap D : \quad u_r = u_\varphi = u_z = 0$$

$$S_2 \cap D : \quad h = 0$$

$$\Gamma_1 \cap D : \begin{cases} h = 0 & \text{(without surface tension)} \\ h = \sigma_p \left(\delta'' - \frac{n^2}{R_1^2}\delta\right), \; \delta(0) = 0 & \text{(with surface tension)} \end{cases}$$

$$\Gamma_2 \cap D : \quad u_r = 0$$

- stability $\Longleftrightarrow \mathrm{Re}(\lambda) < 0$
- $n = 0 \Longrightarrow$ rotationally symmetric flow

Discretization

- approximate finite-dimensional eigenvalue problem obtained by spectral element method (SEM) $\equiv$ hp-finite element method (hp-FEM)
- algorithm implemented in MATLAB $\Longrightarrow$ `eustab`
- generalized eigenvalue problem

$$\mathbf{A}\mathbf{u} = \lambda \mathbf{B}\mathbf{u} \tag{4.1}$$

solved by means of MATLAB functions `eig` and `eigs`

- domain partition: set $\{D^{ij}\}$ of concurrent rectangles

$$D = \bigcup D^{ij}, \qquad i = 1, \ldots, n_r, \;\; j = 1, \ldots, n_z$$

- spectral element
  - domain D^{ij}
  - shape function: polynomial of degree N in both r, z
  - degrees of freedom: values at nodes of product Gauss-Legendre-Lobatto (GLL) quadrature formula of order $2N - 1$
- solution method: Galerkin with GLL numerical integration
- test functions
  - r-momentum eq. : w_r, $= 0$ for $z = 0$ or $r = R_2$
  - φ-momentum eq. : w_φ, $= 0$ for $z = 0$
  - z-momentum eq. : w_z, $= 0$ for $z = 0$
  - continuity eq. : w_p, $= 0$ for $z = L$ or $r = R_1$
  - impermeability eq. : w_δ, $= 0$ for $z = 0$
  - Young-Laplace eq. : w_σ, $= 0$ for $z = L$

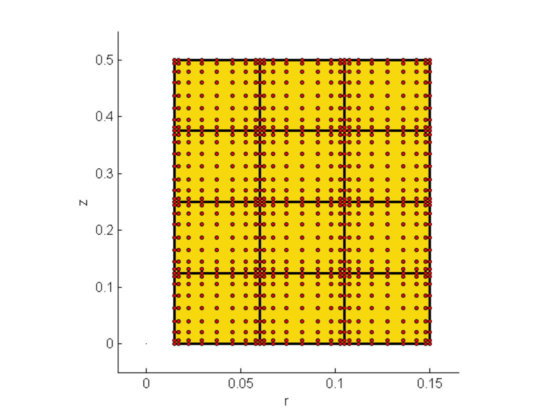

Figure: Net, $n_r = 3$, $n_z = 4$, $N = 8$, 825 nodes.

- solution of the eigenvalue problem

$$\mathbf{A}\mathbf{u} = \lambda \mathbf{B}\mathbf{u}$$

  in MATLAB

- $\mathbf{A}$ is regular, nonhermitian, real for $n = 0$ and complex for $n > 0$
- $\mathbf{B}$ is diagonal and singular: diagonal elements corresponding to continuity equation and Young-Laplace equation are equal to zero, remaining diagonal coefficients are real and positive
- there are $n_\infty = 2n_h + n_\delta$ infinite eigenvalues, where n_h and n_δ is number of degrees of freedom of pressure h and free boundary displacement δ, respectively

- m-function `eig`:
  - based on LAPACK
  - arbitrary $\mathbf{A}$, $\mathbf{B}$
  - all eigenvalues and eigenvectors
  - robust, expensive
- m-function `eigs`:
  - based on ARPACK
  - $\mathbf{A}$ arbitrary, $\mathbf{B}$ hermitian (however, it works in our case)
  - a few eigenvalues and eigenvectors
  - lower computer time and computer memory demands
  - problem: how to specify a set of eigenvalues which we want to find?
    - maximum real part ... impossible due to infinite eigenvalues
    - closest to a specified complex number... but how to choose it?

Following numerical experiments: with m-function `eig`

Results of numerical experiments

- Example 1.
- constant stationary velocities

$$w_{0r} = w_{0\varphi} = 0,\ w_{0z} = C_0, \text{ where}$$

$R_0 = 0.015$	[m]	$\Omega = 5$	[rad $\cdot$ s^{-1}]
$R_2 = 0.15$	[m]	$\varrho = 1000$	[kg $\cdot$ m^{-3}]
$L = 0.5$	[m]	$\sigma_p = 0.073$	[N $\cdot$ m]
$C_0 = 1$	[m $\cdot$ s^{-1}]	$n = 0, 1, 2$	[whole numbers]

- discretization: $n_r = n_z = 1$, $N = 8$
- surface tension influence is insignificant $\Longrightarrow$ presented results computed without surface tension

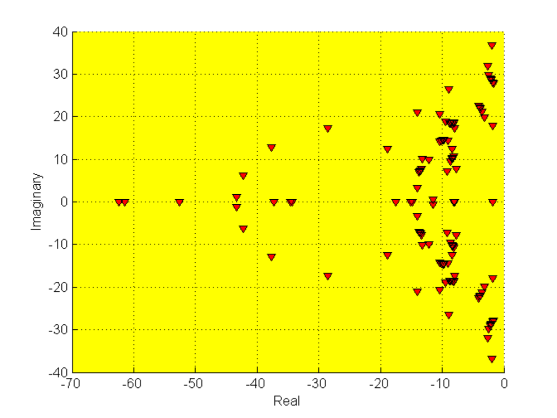

Figure: Ex1-0, eigenvalues, $n = 0$.

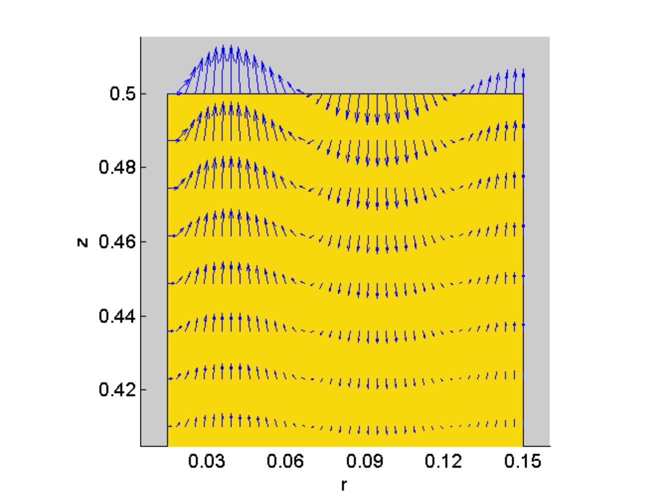

Figure: Ex1-1, real part of the eigenfunction for $n = 0$, $\lambda = -14.91$, meridial section for $\varphi = 0$.

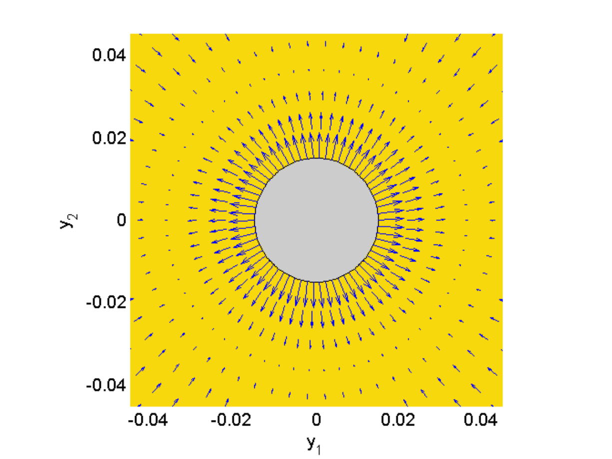

Figure: Ex1-2, real part of the eigenfunction for $n = 0$, $\lambda = -14.91$, normal section for $z = 0.5$.

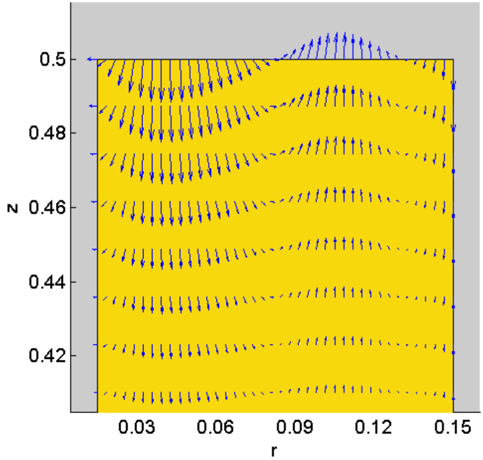

Figure: Ex1-3, real part of the eigenfunction for $n = 1$, $\lambda = -14.91$, meridial section for $\varphi = 0$.

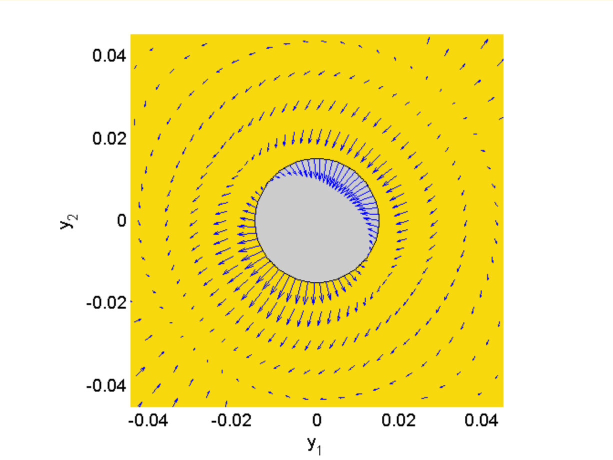

Figure: Ex1-4, real part of the eigenfunction for $n = 1$, $\lambda = -14.91$, normal section for $z = 0.5$.

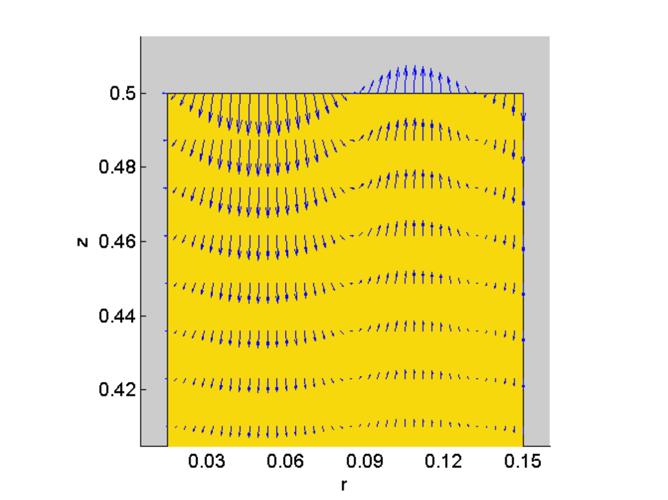

Figure: Ex1-5, real part of the eigenfunction for $n = 2$, $\lambda = -14.88$, meridial section for $\varphi = 0$.

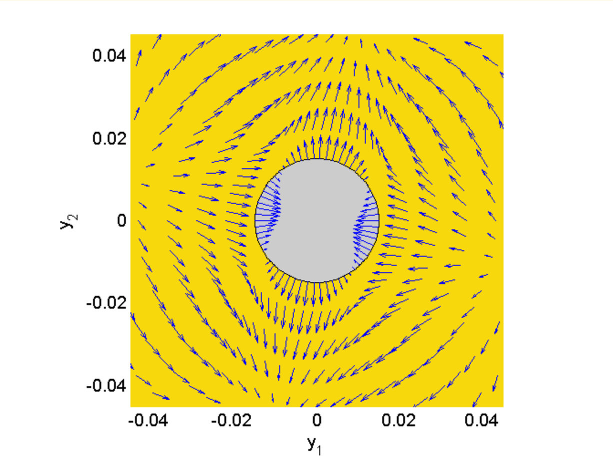

Figure: Ex1-6, real part of the eigenfunction for $n = 2$, $\lambda = -14.88$, normal section for $z = 0.5$.

- 3136 eigenvalues, 1600 finite eigenvalues, $\max \mathrm{Re}(\lambda) \approx -0.04$

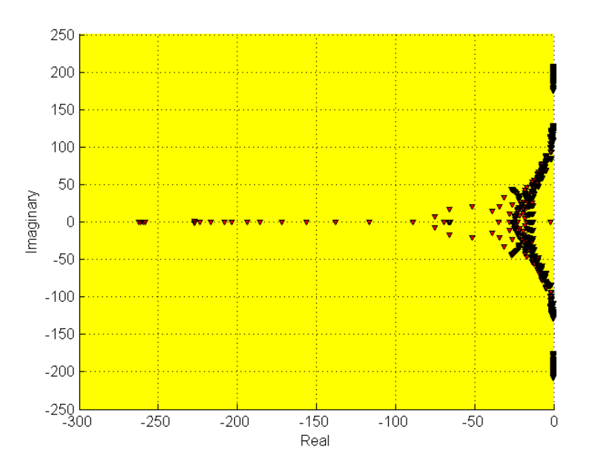

Figure: Ex1-7, eigenvalues, $n_r = 3$, $n_z = 4$, $N = 8$.

- Example 2.
- constant velocities $w_{0r} = 0$, $w_{0z} = C_0$, circumferential velocity

$$c_{0\varphi} = w_{0\varphi}(r) + \Omega r$$

- data as in example 1
- four circumferential velocities in figures Ex2-1, Ex2-2, Ex2-3, Ex2-4
- corresponding maximal real parts of eigenvalues in table Ex2
- computations accomplished for $n_r = 2$, $n_z = 1$, $N = 8$, $n = 0$, without surface tension

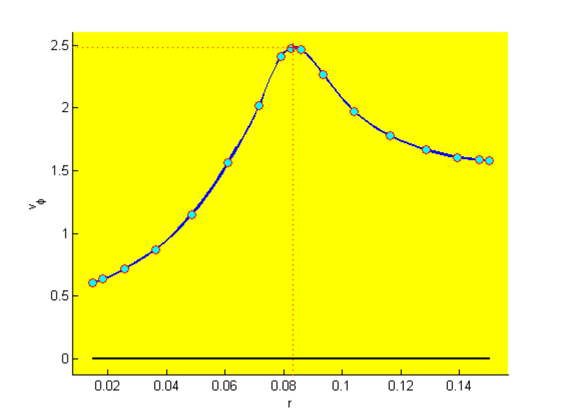

Figure: Ex2-1, $\max c_{0\varphi} = c_{0\varphi}(0.083) = 2.48$

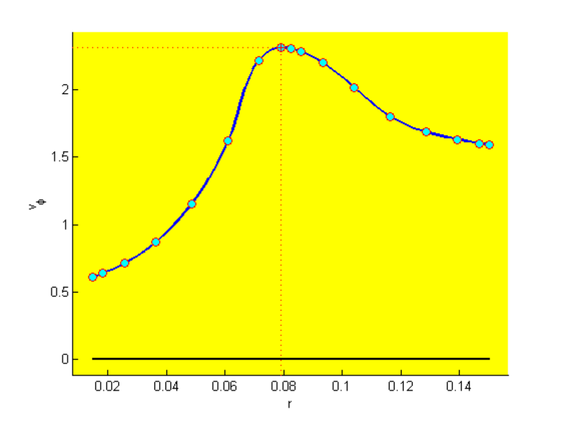

Figure: Ex2-2, $\max c_{0\varphi} = c_{0\varphi}(0.079) = 2.31$

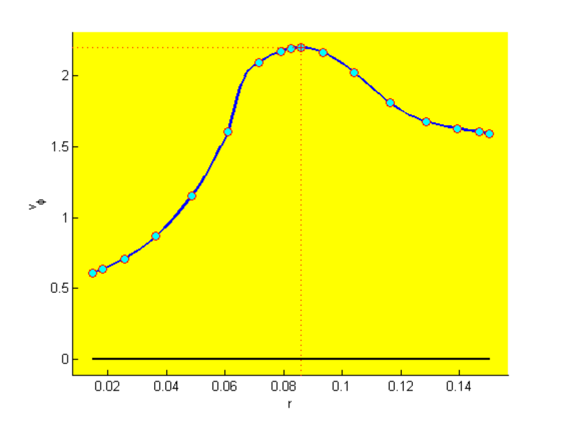

Figure: Ex2-3, $\max c_{0\varphi} = c_{0\varphi}(0.086) = 2.20$

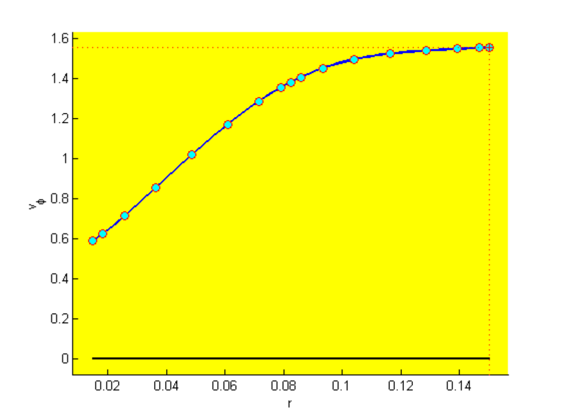

Figure: Ex2-4, $\max c_{0\varphi} = c_{0\varphi}(0.150) = 1.56$

- stability results for circumferential velocities $c_{0\varphi}$ corresponding previous figures

r_{max}	0.083	0.079	0.086	0.150
$c_{0\varphi,max}$	2.48	2.31	2.20	1.56
$\max \mathsf{Re}(\lambda), n = 0$	0.73	−0.03	−0.03	−0.79
$\max \mathsf{Re}(\lambda), n = 1$	2.44	0.16	−0.25	−0.51
$\max \mathsf{Re}(\lambda), n = 2$	2.08	0.57	0.25	−1.62

Table: Ex2, stability for $c_{0\varphi}$ in accordance with figures Ex2-1, Ex2-2, Ex2-3, Ex2-4

- Example 3.
- constant velocities $w_{0r} = 0$, $w_{0z} = C_0$, inflow circumferential velocity

$$w_{0\varphi} = \Omega r(ae^{-r/b} - 1)\,, \quad a = 1.5, b = 0.05,$$

$c_{0\varphi} = w_{0\varphi} + \Omega r$, see figure Ex3-1

- other data

$R_0 = 0.015$	[m]	$C_0 = 1$	$[\mathrm{m} \cdot \mathrm{s}^{-1}]$
$R_2 = 0.15$	[m]	$\Omega = 5$	$[\mathrm{rad} \cdot \mathrm{s}^{-1}]$
$L = 1$	[m]	$\varrho = 1000$	$[\mathrm{kg} \cdot \mathrm{m}^{-3}]$

- computations accomplished for $n_r = 4$, $n_z = 1$, $N = 8$, $n = 0$, without surface tension

- inflow circumferential velocity $c_{0\varphi} = 7.5re^{-r/0.05}$

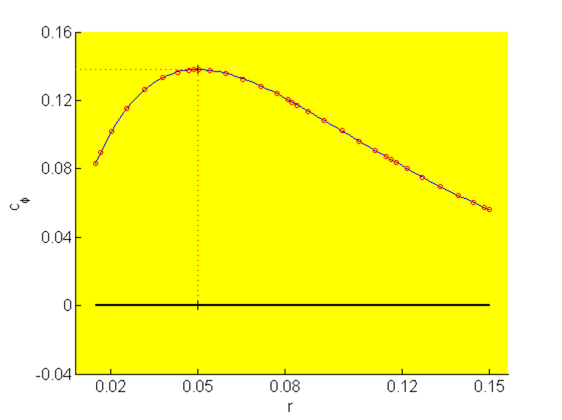

Figure: Ex3-1, $\max c_{0\varphi} = c_{0\varphi}(0.05) = 0.138$

- eigenvalues for $w_{0r} = 0$, $w_{0\varphi} = 5r(1.5e^{-r/0.05} - 1)$, $w_{0z} = 1$

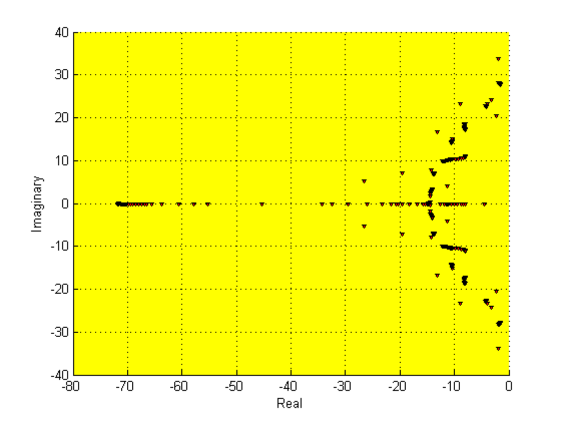

Figure: Ex3-2, $\max \mathrm{Re}(\lambda) = -1.61$

- eigenvalues for flow field computed by FLUENT for boundary conditions

$z = 0: \quad w_r = 0, w_{0\varphi} = 5r(1.5e^{-r/0.05} - 1), w_z = 1$

$z = L: \quad p = p_{cvr}$ (constant pressure invoking cavitating vortex rope)

$r = R_1: \quad p = p_{sv}$ (constant saturated vapour pressure)

$r = R_2: \quad w_r = 0$

see figure Ex3-3

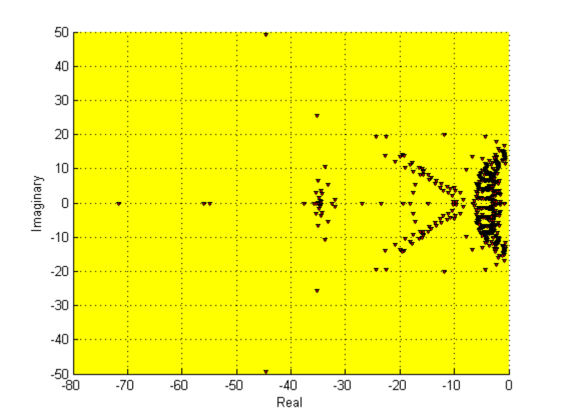

Figure: Ex3-3, $\max \mathrm{Re}(\lambda) = -0.67$

Literature

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